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KALMAN-FILTERING WITH COLOURED MEASUREMENT NOISE FOR DEFORMATION ANALYSIS

Heiner Kuhlmann

Institute for Applications of Geodesy to Engineering, University of Stuttgart
Geschwister-Scholl-Str. 24D, 70174 Stuttgart, Germany

Abstract

Kalman filter is an important tool for deformation analysis combining information on object behaviour and measurement quantities. It is applicable to the four well-known deformation models. Kalman filtering usually requires white measurement and process noise. Due to electronic measurement devices with high sampling rate used nowadays, time dependent systematic deviations arise in neighbouring epochs in a similar way, resulting in auto-correlation. Especially in case of GPS measurements deviations due to multipath and signal propagation are changing slowly, and thus the assumption of white noise is not justified. To eliminate this deficiency within a shaping filter the state vector in Kalman filter is augmented and thus formulating an adequate noise process.

In this paper the modification of Kalman-Filter algorithm with a shaping filter for coloured measurement noise is presented together with an example for deformation analysis using GPS measurements. The differences to an approach neglecting auto-correlation are shown.

1. Kalman-filtering in deformation analysis

Kalman filter is an important tool for deformation analysis combining information on object behaviour and measurement quantities. System and measurement equation are combined in a well-known algorithm for estimating an optimal state vector $\boldsymbol{x}$, containing parameters describing deformation behaviour (Welsch et. al., 2000). Due to recursive algorithm –working from epoch k to k+1- Kalman-filter is applicable in real time. In (1) time discrete system equation is given. Further derivations of (1) will be presented in paragraph 4.

$$\boldsymbol{x}(k+1)=\boldsymbol{T}(k)\cdot\boldsymbol{x}(k)+\boldsymbol{B}(k)\cdot\boldsymbol{u}(k)+\boldsymbol{S}(k)\cdot\boldsymbol{w}(k) \tag{1}$$

with

$\boldsymbol{x}(k)$:	State vector	$\boldsymbol{T}(k)$:	Transition matrix
$\boldsymbol{u}(k)$:	Control input, regulating variable	$\boldsymbol{B}(k)$:	Input coupling matrix
$\boldsymbol{w}(k)$:	System noise	$\boldsymbol{S}(k)$:	System noise coupling

Table 1 shows subdivision of 4 deformation models according to system theory, which are also used in deformation analysis (Welsch, Heunecke, 2001). System equation (1) is derived from the dynamic system. Therefore it corresponds to the dynamic model. It is missing in kinematic model $\boldsymbol{B}(k)\cdot\boldsymbol{u}(k)$, because no reasons for deformation are considered in the model. It is missing in static model $\boldsymbol{T}(k)\cdot\boldsymbol{x}(k)$, because the object is reacting immediately to changes of input. There is no memory. In the identity model there are no causes for deformations, and transitions matrix simplifies to identity matrix $\boldsymbol{I}$.

		Deformations are functions of acting forces	
		no	yes
Deformations are	no	Identity Model	Static Deformation Model
functions of time	yes	Kinematic Deformation Model	Dynamic Deformation Model
		Descriptive Models	Causative Models

Table 1: Classification of deformation models (Kuhlmann, Pelzer, 1997)

Additionally to this state vector $\boldsymbol{x}$ is related linearly to measurements $\boldsymbol{l}$ with design matrix $\boldsymbol{A}$. $\boldsymbol{\varepsilon}$ is the measurement noise:

$$\boldsymbol{l}(k+1) = \boldsymbol{A}(k+1)\cdot\boldsymbol{x}(k+1) + \boldsymbol{\varepsilon}(k+1)\,. \qquad (2)$$

If $\boldsymbol{w}$ and $\boldsymbol{\varepsilon}$ are normally distributed white noise processes state vector estimation is optimal (no bias, minimal variance) (Grewel, Andrews, 1993). Due to electronic measurement devices with high sampling rate used nowadays, time dependent systematic deviations arise in neighbouring epochs in a similar way, resulting in auto-correlation. So white measurement noise is not fulfilled. Time varying correlation of GPS measurements are presented in (Schwieger, 1999). In this paper a strategy to overcome this deficiency in Kalman-filter will be shown.

2. Description of measurements

Figure 1: Measurement of baseline with PDGPS

On the roof of the institute building a short baseline of approximately 7m marked with tripods was built up and observed with Leica® SR530-receiver with AT502-antennas (Figure 1) The sampling rate was 0,2 s and the measurement time 70 min. During measurement the antenna height of one point was changed step by step with a crank every two minutes in steps of 3,2 mm. The baseline was calculated with differential carrier phase solution in post-processing using both frequencies. Figure 2 shows the changing in measured height component and true deformation. The goal is the estimation of the deformation process in a Kalman-filter using the measurements mentioned.

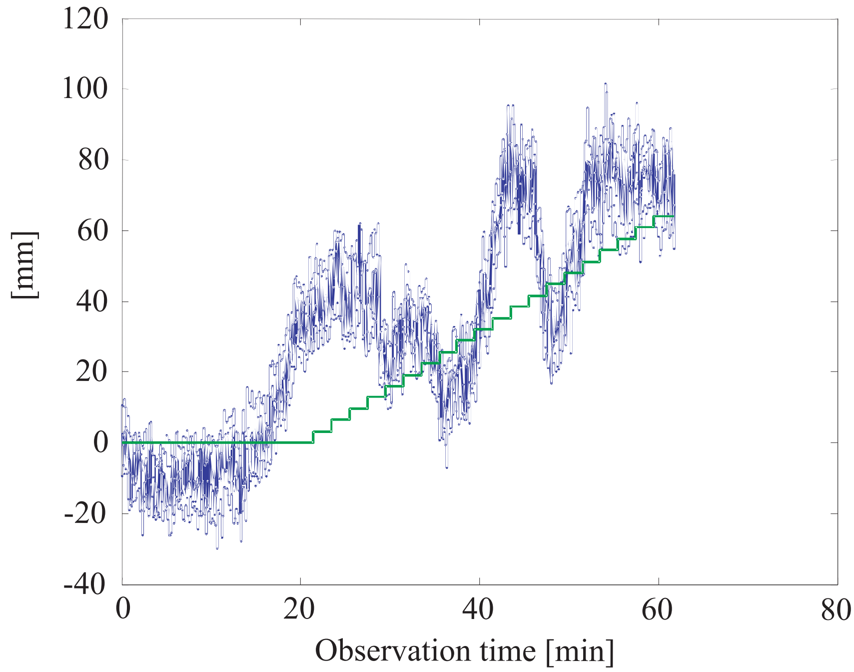

Figure 2: Measurement of changing of height component and true deformation

As it will be shown later the stochastic model of observations is essential for the deformation model in Kalman-filter. Thus the same baseline was observed in the same way but both antennas were fixed. The sampling rate was 1 s and the measurement time 7 hours. Figure 3 shows the result of $n = 26000$ observations. Observation deviations are rising up to 10 cm, mainly resulting from multipath (Schwieger, 1999) or (Georgiadou, Kleusberg, 1988).

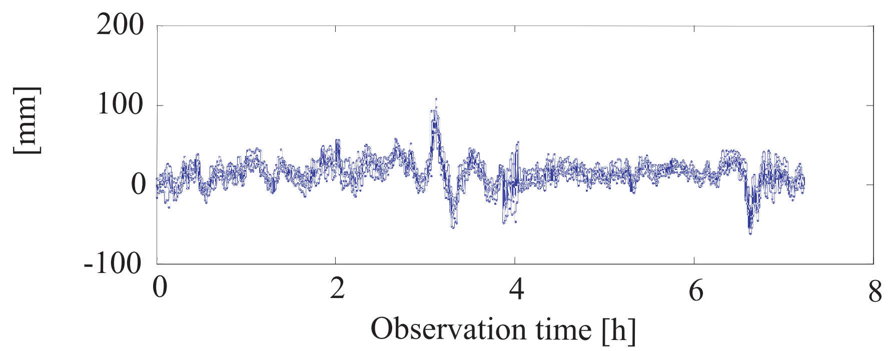

Figure 3: Determination of stochastic model: changing of ellipsoidal height of fixed rover station

3. Determination of stochastic model

3.1. Elementary error model

Deviations of measurements can be sub-divided in elementary errors, each with the expectation value zero. Some of these errors arise in a systematic manner, because they influence observations taken in an other time or location in a similar way (correlating errors). The others vary from time to time in a random way (non-correlating errors) (Schwieger, 1999). One example for correlating errors in case of GPS-measurements is the multipath effect. The deviation in pseudo-range is almost identical between two observations with a time difference

of some tenth of a second. Other examples are propagation errors in troposphere and orbit errors. The magnitude of the sum of non-correlating errors and correlating errors can be described by standard deviations σ_δ and σ_Δ . The latter one also needs an information about changing in time to describe the stochastic model of observations.
Following (Schwieger, 1999) it can be stated that GPS measurement deviations are correlated. In case of short baselines and short measurement periods deviations are following a Gauß-Markov-process with correlation function

$$C_l(t) = e^{-a|t|} \quad . \tag{3}$$

Thus the stochastic model of the above specified measurements is given by

$$\underset{n,n}{\boldsymbol{\Sigma_{ll}}} = \begin{vmatrix} \sigma_\delta^2 + \sigma_\Delta^2 \quad e^{-\alpha\cdot 0} & \sigma_\Delta^2 \cdot e^{-\alpha\cdot 1} & \cdots & \sigma_\Delta^2 \cdot e^{-\alpha\cdot(n-1)} \\ \sigma_\Delta^2 \cdot e^{-\alpha\cdot 1} & \sigma_\Delta^2 + \sigma_\Delta^2 \cdot e^{-\alpha\cdot 0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \sigma_\Delta^2 \cdot e^{-\alpha\cdot 1} \\ \sigma_\Delta^2 \cdot e^{-\alpha\cdot(n-1)} & \vdots & \sigma_\Delta^2 \cdot e^{-a\cdot 1} & \sigma_\delta^2 + \sigma_\Delta^2 \cdot e^{-\alpha\cdot 0} \end{vmatrix} \tag{4}$$

3.2. Parameter estimation of stochastic model

Assuming a Gaussian white noise process with variance s^2 the covariance matrix of observations is $\Sigma_{ll} = \sigma^2 \cdot \boldsymbol{I}$. Then mean value of n numbers and corresponding variance is given by

$$\bar{x} = \frac{1}{n} \cdot e^T \cdot l \quad , \quad \sigma_{\bar{x}}^2 = \frac{e^T \cdot \Sigma_{ll} \cdot e}{n^2} = \frac{n \cdot \sigma^2}{n^2} \text{ with } e^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} . \tag{5}$$

As well known the variance is reduced by n. The numerator in (5) is the sum of elements of Σ_{ll}. In case of coloured measurement noise (4) this leads to

$$\sigma_{\bar{x}}^2 = \frac{1}{n} \cdot \sigma_\delta^2 + \frac{1}{n} \cdot \sigma_\Delta^2 + \frac{2}{n^2} \sum_{k=1}^{n-1} (n-k) \cdot \sigma_\Delta^2 \cdot e^{-\alpha k} . \tag{6}$$

Taking the observation vector of Figure 3 new samples can be generated by calculating mean values of different numbers m. For instance for $m = 5$ this leads to a sample with 5200 elements. In each new sample with different m, a second mean value and with residuals to this mean value an empirical variance can be calculated which measures the theoretical variance Figure 4 for averaged observations.
Figure 2 shows the numbers. It can be seen that the standard deviation is decreasing with rising m but not as much as for white noise, indicating the phenomena of auto-correlation.

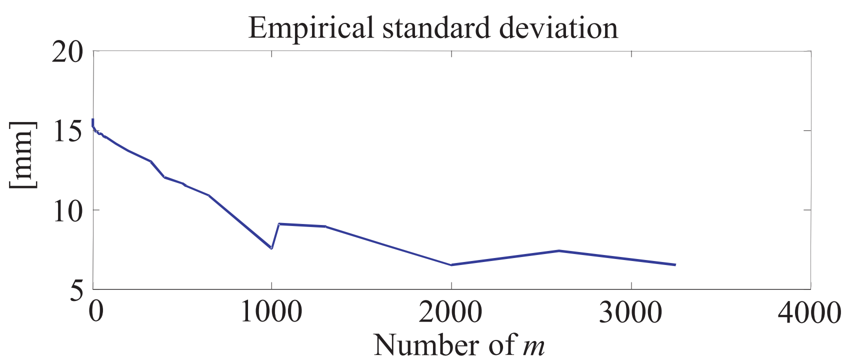

Figure 2: Empirical standard deviation from mean value of m observations

The unknown parameters of the stochastic model σ_δ, σ_Δ and α in (6) can be estimated by adjustment by observation equations with these “observed” variances. In the stochastic model of this adjustment it has to be introduced that the empirical variances are functionally correlated and that there are different degrees of freedom. Due to the high degree of freedom the theoretical χ^2-distributuion of variances is neglected (Niemeier, 2002). Hence variance estimation is rough.

	Estimated value	Standard deviation
$\sigma\delta$	4.3 mm	0.92 mm
$\sigma\Delta$	12.5 mm	0.84 mm
α	0.005 1/s	0.0010 1/s

Table 2: Estimated parameter of stochastic model

To give a better impression about this numbers, table 3 shows the correlation coefficient for different time differences between two observations. There is high auto-correlation, numerical values of long term movement of measurement deviations (Figure 2). It can also be seen that the assumption of white noise is fulfilled in practice for time differences of more than 5 minutes.

Δt	1 sec	5 sec	30 sec	1 min	2 min	5 min	10 min
ρ	0.89	0.87	0.77	0.66	0.49	0.20	0.04

Table 3: Correlation coefficient of two GPS-observations for different time differences

4. State vector augmentation, shaping filter

As shown there is coloured measurement noise, thus prerequisite of white noise for Kalman-Filter is not fulfilled. For the introduction of correlated noise in a Kalman-Filter a shaping filter is necessary (Schrick, 1977). Basic idea is the formulation of a second process in the system equation besides the deformation process. This second process is a linear dynamic system and excited by a white noise process. The output is the coloured noise process of the measurement (Huep, 1986).

For better understanding it is useful to know that system equation (1) is a state space representation of a dynamic system. If ordinary differential equations are given describing the behaviour of the object, introduction of state variables leads to an ordinary vector differential equation of first order. (1) is the piecewise solution of this equation (Isermann, 1988).

Continuing this approach a representation of a shaping filter as an ordinary vector differential equation of first order is necessary. Then both dynamic systems can be combined in one equation where the homogenous part is presented in (7). The index 1 is indicating the deformation process and 2 the shaping filter. ***F*** is called system matrix.

$$\begin{vmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \end{vmatrix} = \begin{vmatrix} \boldsymbol{F}_1 & 0 \\ 0 & \boldsymbol{F}_2 \end{vmatrix} \cdot \begin{vmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{vmatrix} \tag{7}$$

To get a time discrete solution as in (1) piecewise solution is necessary. In case of coloured measurement noise both differential equation are independent so they can be solved separately. For coloured system noise, not treated in this paper this procedure is not valid.

For a Gauss-Markov-Process with auto-correlation function (3) the following differential equation representing a linear dynamic system is suitable (Huep, 1986)

$$\dot{x}_2(t) = -\alpha \cdot x_2(t) + w_2'(t)\,. \tag{8}$$

Piecewise solution leads to the following equation as second part of the system equation

$$x_2(k+1) = e^{-\alpha\cdot\Delta t} \cdot x_2(k) + e^{-\alpha\cdot\Delta t} \cdot w_2(k) \tag{9}$$

where the state variable x_2 describes the long term movement of correlating measurement deviations. The system noise w_2 is a white noise process with variance σ_{Δ}^2.

5. Results and conclusion

For the above described deformation a random-walk-process (identity model) or a kinematic model is possible, since there are no acting forces available. In case of identity model system noise w_1 should correspond to antenna motion. According to (Pelzer, 1987) system noise should be of the same magnitude as the deformation between two epochs (Δt = 0,2 s). Here the antenna height was manipulated by a crank in steps of 3,2 mm which took about 0,3 s and so σ_{w1} = 2 mm was chosen. This leads to the following system equation:

$$\begin{vmatrix} x_1(k+1) \\ x_2(k+1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & e^{-\alpha\cdot\Delta t} \end{vmatrix} \cdot \begin{vmatrix} x_1(k) \\ x_2(k) \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & e^{-\alpha\cdot\Delta t} \end{vmatrix} \cdot \begin{vmatrix} w_1(k) \\ w_2(k) \end{vmatrix}$$
$$\boldsymbol{x}(k+1) = \boldsymbol{T}(k) \cdot \boldsymbol{x}(k) + \boldsymbol{S}(k) \cdot \boldsymbol{w}(k) \tag{10}$$

The observed changing of height l has to match with antenna motion x_1 together with the state variable of the shaping filter x_2. ε contents non-correlating measurement deviations with variance σ_{δ}^2. This leads to the following measurement equation:

$$l(k+1) = \begin{vmatrix} 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} x_1(k+1) \\ x_2(k+1) \end{vmatrix} + \varepsilon(k+1) \quad = \quad \mathbf{A} \cdot \mathbf{x}(k+1) + \boldsymbol{\varepsilon}(k+1)\,. \tag{11}$$

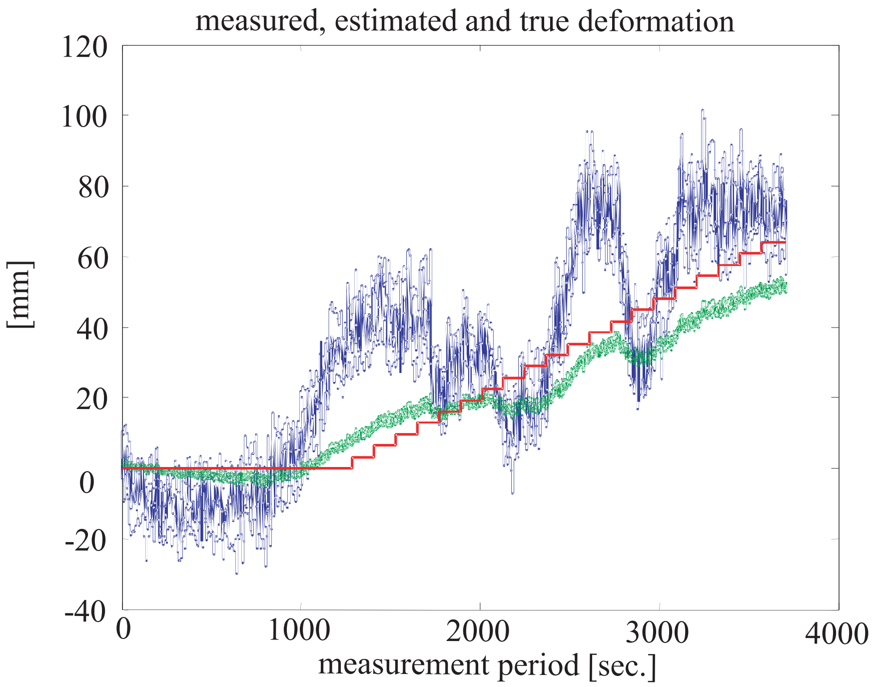

Figure 5: Parameter estimation in Kalman-filter (identity model) with shaping filter

Figure 5 shows the result of the deformation analysis. Estimation follows deformation in general, but there a still some long term deviations. Measurement noise is not completely reduced. During the procedure no significant innovation –difference between prediction with system equation and measurement equation- was detected. That means that the chosen functional and stochastic models correspond with reality. For statistical test of innovation see (Welsch et. al., 2000).

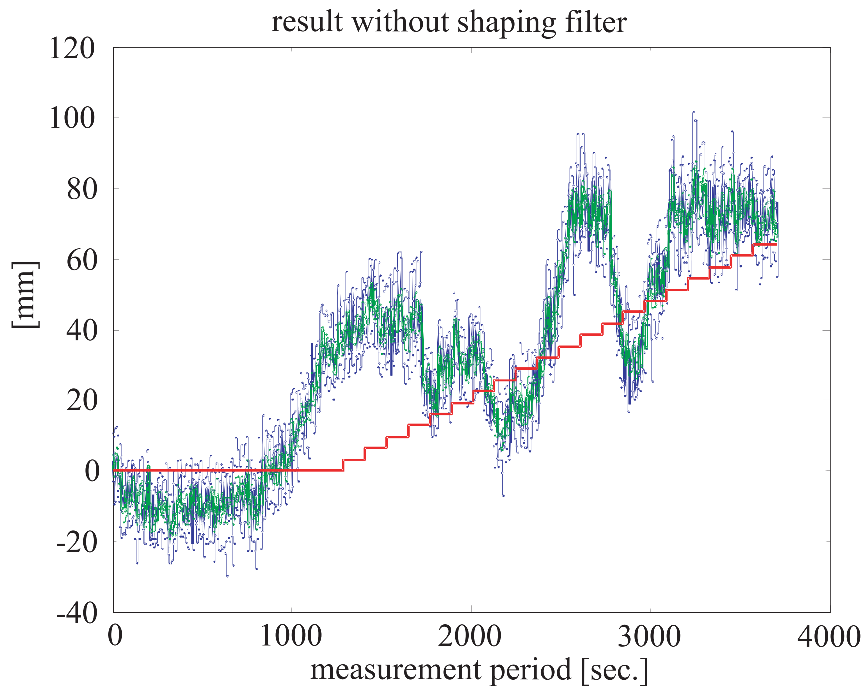

Figure 6: Parameter estimation in Kalman-filter (identity model) without shaping filter

To emphasise the quality of deformation estimation Figure 6 shows the result without shaping filter. Here the system equation is the first line of (10) and measurement equation is

$$l(k+1) = x_1(k+1) + \varepsilon(k+1) \quad = \quad \mathbf{A} \cdot \mathbf{x}(k+1) + \boldsymbol{\varepsilon}(k+1) \;. \qquad (12)$$

Variance of measurement noise is –non optimal- $\sigma_\varepsilon^2 = \sigma_\Delta^2 + \sigma_\delta^2$. The estimated deformation is a little bit smoother than the measurement but follows it in general. Any changing of σ_ε does not change this, only the smoothing effect varies. That shows that in this case it is not possible to formulate an adequate deformation model without shaping filter. Since there is no state variable which can absorb long term variation of measurement deviations estimation follows measurement.

As a conclusion following statements can be made:

- For measurements with high sampling rate often auto-correlation occur.
- For GPS-measurements auto-correlation is significant.
- In case of auto-correlation standard deviation is not sufficient for stochastic model. Parameters describing temporal behaviour of correlation are necessary.
- State vector augmentation in Kalman-filter is suitable if coloured noise is used.
- Neglecting auto-correlation in Kalman-filter leads to parameter estimation not corresponding to motion of object.

5. References

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